STABILITY OF A GROWING VISCOELASTIC REINFORCED ROD SUBJECTED TO AGING

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The stability of growing nonuniformly aging viscoelastic reinforced rods is investigated. The equations of state of a viscoelastic material are described by the equations of viscoelasticity theory of nonuniformly aging bodies [1], and the strains and stresses in the reinforcement are related to each other by Hooke's law. The rod is acted on it its own weight and a concentrated force which varies in time.

The stability condition of the rod is obtained on a semi-infinite time interval. The determination of stability corresponds to the determination of the stability of dynamical systems in the Lyapunov sense. The problem of stability of a rod on a finite time interval has been investigated. The adopted formulation of the stability problem on a finite time interval is based on investigations of the stability of dynamical systems of N. G. Chetaev.

The stability of nonuniformly aging reinforced viscoelastic rods in the case in which the transverse cross section and length of the rod remained constant in the deformation process has been studied in [2, 3].

1. Model of a Growing Reinforced Viscoelastic Rod Subjected to Aging. We shall consider a hollow rod which grows both in the longitudinal, i.e, along the rod axis, and in the transverse direction. For the sake of simplicity we shall assume that the transverse cross section of the rod has two symmetry axes; the reinforcement is distributed symmetrically with respect to these axes. We shall assume without loss of generality that the initial length of the rod l_0 is equal to zero.

Let the variation law of the rod length in time $l(t)$ be specified; $l(t)$ is a bounded piecewise-continuous monotonically increasing function. We shall denote the time at the conclusion of which the length of the growing rod reaches the value s as $\tau_1^*(s)$. It is evident that $\tau_1^*(s) = l^{-1}(s)$. Here l^{-1} is the inverse function to l.

We shall assume that the length of the reinforcing elements at the time under discussion is equal to the rod length $l(t)$. Actually, the length of the reinforcing elements can exceed the length of the main material, as often happens, for example, in the fabrication of tall iron-concrete columns. In case of necessity this peculiarity can be taken into account in setting up the equations which describe the deformation process of the rod.

The kinematics of the rod growth in the transverse direction can be diverse. Two of its possible mechanisms have been discussed in detail in [4]. Assuming this or the other growth mechanism, one can determine the time of creation of the main material $\tau^*(\rho)$ in the neighborhood of the point with the coordinates $\rho = \{x, y, s\}$.

Starting from the time T_0, the formation of a body which for simplicity's sake we shall consider to be absolutely rigid (a disk) occurs on the end of the rod. We shall denote the height of the disk up to time t by $Z(t)$.

The equation of state for a nonuniformly aging viscoelastic material in a uniaxial stress state shall be taken in the form [1]

$$\sigma(\rho, t) = E(t - \tau^*(\rho))\,\varepsilon(\rho, t) - \int\limits_{\tau^*(\rho)}^{t} R(t - \tau^*(\rho), \tau - \tau^*(\rho))\,\varepsilon(\rho, \tau)\,d\tau,$$

where σ and ε are the stress and strain in the growing rod, $E(t)$ is the modulus of the elastically instantaneous strains, and $R(t, \tau)$ is the relaxation kernel of the aging viscoelastic material of the rod.

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The stresses σ_r and strains ε_r in the reinforcement satisfy Hooke's law, i.e., the equality

$$\sigma_r(t, \rho) = E_r \varepsilon_r(t, \rho)$$

holds, where E_r is the modulus of elasticity of the reinforcing material.

2. Equation of Motion of a Growing Reinforced Viscoelastic Rod. The process of deformation of a growing viscoelastic rod in which its axis maintains a rectilinear vertical position can be treated as slow motion. We shall call it the unperturbed motion. Next we shall assume that in the absence of external loads and under the condition of weightlessness of the rod its axis has an initial curve (an initial perturbation) in the yOs plane which is describable by the function $w_0 = w_0(s)$. Under the action of its own weight and the weight of the heavy disk the rod receives an additional deflection (the desired perturbation) $w = w(t, s)$ which depends on the coordinate s and the time t. As a result the rod elements are displaced not only in the vertical but also in the horizontal direction. We shall call such motion of the rod perturbed.

The bending moment acting in the yOs plane in the cross section of the rod with the coordinate s at time t is equal to

$$M(t, s) = \int_{F_r(s)} E_r \varepsilon_r(t, \rho)\, y dF + \int_{F(t,s)} \Bigg[F(t - \tau^*(\rho))\varepsilon(\rho, t) - \int_{\tau^*(\rho)}^{t} R(t - \tau^*(\rho), \tau - \tau^*(\rho))\, \varepsilon(\rho, \tau)\, d\tau\Bigg] y dF. \tag{2.1}$$

Here F(t, s) and $F_r(s)$ are the areas of the transverse cross section of the viscoelastic material and the reinforcement in the cross section with coordinate s at time t.

We shall assume when finding the strains that the transverse cross section of the rod, whose position at time $\tau_1^*(s)$ was determined by a flat inner contour perpendicular to the curved axis of the rod, remains flat and perpendicular to the curved axis of the rod also at the time t. In other words, the particles created at the time $\tau^*(\rho)$ and which appeared in a single plane perpendicular to the curved rod axis with the other points created in the interval $(\tau_1^*(s), \tau^*(\rho))$ remain with them in the same plane at any time t, and this plane is perpendicular to the curved rod axis.

Due to this hypothesis we have for the strain the expression

$$\varepsilon(\rho, t) = \Delta\varkappa(t, s)y, \tag{2.2}$$

where $\Delta\varkappa(t, s) = \varkappa(t, s) - \varkappa(\tau^*(\rho), s)$, and

$$\varkappa(t, s) = \frac{\partial^2}{\partial s^2}[w(t, s) + w_0(s)]\left\{1 - \left[\frac{\partial w(t, s)}{\partial s} + \frac{\partial w_0(s)}{\partial s}\right]^2\right\}^{-1/2} - \frac{\partial^2 w_0(s)}{\partial s^2}\left\{1 - \left[\frac{\partial w_0(s)}{\partial s}\right]^2\right\}^{-1/2} \tag{2.3}$$

is the variation of the rod curvature at time t in comparison with the initial curvature in the cross section with coordinate s.

If the deflection of the rod $w(t, s) + w_0(s)$ is small, i.e., one can neglect the quantity $\left\{\frac{\partial}{\partial s}[w(t, s) + w_0(s)]\right\}$ in comparison with unity, then

$$\varkappa(t, s) = \partial^2 w(t, s)/\partial s^2 \tag{2.4}$$

and the dependence (2.1) with the expressions (2.2) and (2.3) taken into account will take the form

$$M(t, s) = E_r J_r(s)\, \Delta\varkappa(t, s) + EJ(t, s)\, \varkappa(t, s) - \int_{F(t,s)} E(t - \tau^*(\rho))\, \varkappa(\tau^*(\rho), s) y^2 dF - \int_{F(t,s)} \int_{\tau^*(\rho)} R(t - \tau^*(\rho), \tau - \tau^*(\rho))\, \varkappa(\tau, s)\, d\tau y^2 \times dF + \int_{F(t,s)} \int_{\tau^*(\rho)}^{t} R(t - \tau^*(\rho), \tau - \tau^*(\rho))\, \varkappa(\tau^*(\rho), s)\, d\tau y^2 dF, \tag{2.5}$$

where

$$J_{\mathrm{I}}(s)=\int\limits_{F_{\mathrm{I}}(s)} y^2dF, \quad EJ(t, s)=\int\limits_{F(t,s)} E(t-\tau^*(\rho))\, y^2dF.$$

As is well known,

$$R(t, \tau)=\partial L(t, \tau)/\partial\tau, \tag{2.6}$$

where $L(t, \tau) = E(\tau) - Q(t, \tau)$, $E(\tau)$ is the modulus of the elastically instantaneous strain, and $Q(t, \tau)$ is the relaxation measure, with $Q(t, t) = 0$.

Let us transform the last term in Eq. (2.5) with the relationship (2.6) taken into account

$$\int\limits_{F(t,s)}\int\limits_{\tau^*(\rho)}^{t} R(t-\tau^*(\rho), \tau-\tau^*(\rho))\, \varkappa(\tau^*(\rho), s)\, d\tau y^2dF=\int\limits_{F(t,s)} \varkappa(\tau^*(\rho), s)\,[E(t-\tau^*(\rho))-L(t-\tau^*(\rho), 0)]\, y^2dF.$$

Then one can represent the expression (2.5) in the form

$$M(t, s)=E_{\mathrm{I}}J_{\mathrm{I}}(s)\,\Delta\varkappa(t, s)+EJ(t, s)\,\varkappa(t, s)-$$
$$-\int\limits_{F(t,s)}\left[L(t-\tau^*(\rho), 0)\,\varkappa(\tau^*(\rho), s)-\int\limits_{\tau^*(\rho)}^{t} R(t-\tau^*(\rho), \tau-\tau^*(\rho))\,\varkappa(\tau, s)\, d\tau\right] y^2dF.$$

Taking into account the specific mechanism of growth of the rod, one can change the integration order in the double integral

$$\int\limits_{\tau_1^*(s)}^{t}\int\limits_{\tau^*(\rho)}^{t} R(t-\tau^*(\rho), \tau-\tau^*(\rho))\,\varkappa(\tau, s)\, d\tau y^2(\tau^*(\rho))\, dF(\tau^*(\rho))=$$
$$=\int\limits_{\tau_1^*(s)}^{t} \widetilde{R}(t, \tau)\,\varkappa(\tau, s)\, d\tau,$$

where

$$\widetilde{R}(t, \tau)=\int\limits_{\tau_1^*(s)}^{\tau} R(t-\tau^*(\rho), \tau-\tau^*(\rho))\, y^2(\tau^*(\rho))\, dF(\tau^*(\rho)).$$

One can call the function $\widetilde{R}(t, \tau)$ the reduced relaxation kernel of the viscoelastic rod material in the cross section with the coordinate s.

We find by direct substitution that the equality

$$M(t, s)=E_{\mathrm{I}}J_{\mathrm{I}}(s)\,\Delta\varkappa(t, s)+\widetilde{L}(t, t, s)\,\varkappa(t, s)-\int\limits_{\tau_1^*(s)}^{t}\frac{\partial\widetilde{L}(t, \tau, s)}{\partial\tau}\,\varkappa(\tau, s)\, d\tau, \tag{2.7}$$

occurs for a viscoelastic reinfored rod, where

$$\widetilde{L}(t, \tau, s)=\int\limits_{\tau_1^*(s)}^{\tau} L(t-\xi, \tau-\xi)\, y^2(\xi)\, dF(\xi).$$

Next we shall write the equation of quasistatic equilibrium of the rod

$$\frac{\partial^2 M(t, s)}{\partial s^2}=-\frac{\partial}{\partial s}\left\{N(t, s)\frac{\partial}{\partial s}[w(t, s)+w_0(s)]\right\}, \tag{2.8}$$

where

$$N(t, s) = \int_{s}^{l(t)} \int_{F(t,z)} p(\rho') \, dF dz + qZ(t),$$

$p(\rho')$ is the specific weight of the rod material, $\rho' = \{x, y, z\}$, q is the weight of unit length of the disk, and $Z(t) \equiv 0$ for $t \leqslant T_0$.

If the upper end of the rod is free, Eq. (2.8) will take the form

$$\frac{\partial}{\partial s}\left[E_{\mathrm{I}} J_{\mathrm{I}}(s) \Delta \varkappa(t, s) + \widetilde{L}(t, t, s) \varkappa(t, s) - \int_{\tau_1^*(s)}^{t} \frac{\partial \widetilde{L}(t, \tau, s)}{\partial \tau} \varkappa(\tau, s) \, d\tau\right] = -N(t, s) \frac{\partial}{\partial s}[w(t, s) + w_0(s)]. \tag{2.9}$$

In the particular case in which growth of the rod occurs only in the direction of the s axis, Eq. (2.9) is transformed to the form

$$\frac{\partial}{\partial s}\left[E_{\mathrm{I}} J_{\mathrm{I}}(s) \Delta \varkappa(t, s) + E\left(t - \tau_1^*(s)\right) J(s) \varkappa(t, s) - J(s) \int_{\tau_1^*(s)}^{t} R\left(t - \tau_1^*(s), \tau - \tau_1^*(s)\right) \varkappa(\tau, s) \, d\tau - J(s) L\left(t - \tau_1^*(s), 0\right) \varkappa\left(\tau_1^*(s), s\right)\right] = -N(t, s) \frac{\partial}{\partial s}[w(t, s) + w_0(s)]. \tag{2.10}$$

The solutions of Eqs. (2.9) and (2.10) in the case of sealing of the lower end should satisfy the following boundary conditions:

$$s = 0: \quad w(t, s) = \frac{\partial w(t, s)}{\partial s} = 0, \tag{2.11}$$

$$s = l(t): \quad M(t, s) = \begin{cases} 0, & 0 \leqslant t < T_0, \\ \frac{1}{2} qZ^2(t) \frac{\partial}{\partial s}[w(t, s) + w_0(s)], & t > T_0. \end{cases}$$

The integrodifferential equation (2.9) obtained is the equation of the perturbed motion of a reinforced nonuniformly aging viscoelastic growing rod.

<u>3. Variational Formulation of the Problem of the Longitudinal Deflection of a Growing Reinforced Viscoelastic Rod.</u> Variational methods [5] prove to be advisable sometimes in obtaining the solution of viscoelasticity problems. They are especially effective in those cases in which the Volterra principle is inapplicable. The problem of the accretion of viscoelastic bodies (in particular, the problem under discussion here of a growing rod) is one of these examples.

On the basis of the Lagrange principle the work of the internal forces δU in the body by possible displacements which are in agreement with the geometric boundary conditions should be equal to the work of the external forces δA by those same displacements:

$$\delta U = \delta A. \tag{3.1}$$

Considering variations of the instantaneous values of the displacements, and consequently also the strains $\delta\varepsilon(t, \rho)$, we write

$$\delta U = \int_{V_{\mathrm{I}}(t)+V(t)} \sigma(t, \rho) \, \delta\varepsilon(t, \rho) \, dV$$

or

$$\delta U = \int_{V_{\mathrm{I}}(t)} E_{\mathrm{I}} \varepsilon_{\mathrm{I}}(t, \rho) \, \delta\varepsilon(t, \rho) \, dV + \int_{V(t)} \left[E(t - \tau^*(\rho)) \, \varepsilon(t, \rho) - \int_{\tau^*(\rho)}^{t} R(t - \tau^*(\rho), \tau - \tau^*(\rho)) \, \varepsilon(\tau, \rho) \, d\tau\right] \delta\varepsilon(t, \rho) \, dV, \tag{3.2}$$

where $V(t)$ is the volume occupied by the viscoelastic material and $V_I(t)$ is the volume occupied by the reinforcement.

Let us introduce the potentials

$$u_0 = \frac{1}{2} E(t - \tau^*(\rho))\, \varepsilon^2(t, \rho), \quad u_I = \frac{1}{2} E_I \varepsilon_I^2(t, \rho),$$

$$u_1 = -\varepsilon(t, \rho) \int_{\tau^*(\rho)}^{t} R(t - \tau^*(\rho), \tau - \tau^*(\rho))\, \varepsilon(\tau, \rho)\, d\tau.$$

From this follows

$$\sigma(t, \rho) = \frac{\partial}{\partial \varepsilon(t, \rho)} (u_0 + u_1), \quad \sigma_I(t, \rho) = \frac{\partial u_I}{\partial \varepsilon_I(t, \rho)}.$$

Then

$$\delta U = \int_{V(t)} \delta\varepsilon(t, \rho) \frac{\partial}{\partial \varepsilon(t, \rho)} (u_0 + u_1)\, dV + \int_{V_I(t)} \delta\varepsilon(t, \rho) \frac{\partial u_I}{\partial \varepsilon_I(t, \rho)}\, dV.$$

Determining the variation of the strains $\delta\varepsilon(t, \rho)$ by the expression

$$\delta\varepsilon(t, \rho) = \delta\varkappa(t, s) y, \quad \delta\varkappa(t, s) \approx \partial^2 \delta w(t, s)/\partial s^2,$$

we represent Eq. (3.2) in the form

$$\delta U = \int_0^{l(t)} \delta\varkappa(t, s) \int_{F_I(s)} E_I \varepsilon_I(t, \rho)\, y\, dF ds + \int_0^{l(t)} \delta\varkappa(t, s) \int_{F(t,s)} \Big[E(t - \tau^*(\rho))\, \varepsilon(t, \rho) - \int_{\tau^*(\rho)}^{t} R(t - \tau^*(\rho), \tau - \tau^*(\rho))\, \varepsilon(\tau, \rho)\, d\tau \Big] y\, dF ds \tag{3.3}$$

or

$$\delta U = \int_0^{l(t)} M(t, s)\, \delta\varkappa(t, s)\, ds.$$

It is evident that Eq. (3.1) is the minimality condition of the function $\mathcal{E} = U - A$, where A is the work of the external forces and $U = \int_{V_I(t)} u_I dV + \int_{V(t)} (u_0 + u_1)\, dV.$

Considering a rod for which the boundary conditions (2.11) are satisfied, we find the work A of the external forces which is done by them upon deformation of the rod. It is made up of the work of the distributed forces of its own weight and the weight of the disk. Restricting ourselves to the case of small displacements, we obtain

$$A(t) = \frac{1}{2} \int_{V_I(t)+V(t)} p(\rho) \int_0^s \left\{ \left[\frac{\partial w(t, \xi)}{\partial \xi} + \frac{\partial w_0(\xi)}{\partial \xi} \right]^2 - \left[\frac{\partial w(\tau^*(\rho'), \xi)}{\partial \xi} + \frac{\partial w_0(\xi)}{\partial \xi} \right]^2 \right\} d\xi dV +$$
$$+ q \int_0^{Z(t)} \left\{ - \int_0^{l(t)} \left[1 - \frac{1}{2} \left(\frac{\partial w(t, \xi)}{\partial \xi} + \frac{\partial w_0(\xi)}{\partial \xi} \right)^2 \right] d\xi + \right.$$
$$+ \int_0^{l(\tau^*(Z))} \left[1 - \frac{1}{2} \left(\frac{\partial w(\tau^*(Z), \xi)}{\partial \xi} + \frac{\partial w_0(\xi)}{\partial \xi} \right)^2 \right] d\xi + \frac{Z}{2} \left[\left(\frac{\partial w(t, s)}{\partial s} + \frac{\partial w_0(s)}{\partial s} \right)^2 \bigg|_{s=l(t)} - \right.$$
$$\left. \left. - \left(\frac{\partial w(\tau^*(Z), s)}{\partial s} + \frac{\partial w_0(s)}{\partial s} \right)^2 \bigg|_{s=l(\tau^*(Z))} \right] \right\} dZ,$$

where $\tau^*(Z)$ is the time of creation of the elements of the rigid disk with the coordinate Z and $\tau^*(\rho')$ is the time of creation of the rod element with the coordinates $\rho' = \{x, y, \xi\}$. It is assumed here that $Z(t) \equiv 0$ for $t \leqslant T_0$.

The variation of the work of the external forces is equal to

$$\delta A(t) = \int\limits_{V_\Gamma(t)+V(t)} p(\rho) \int\limits_0^s \left[\frac{\partial w(t,\xi)}{\partial \xi} + \frac{\partial w_0(\xi)}{\partial \xi}\right] \frac{\partial \delta w(t,\xi)}{\partial \xi} d\xi dV + \tag{3.4}$$

$$+ q \int\limits_0^{Z(t)} \left\{ \int\limits_0^{l(t)} \left[\frac{\partial w(t,\xi)}{\partial \xi} + \frac{\partial w_0(\xi)}{\partial \xi}\right] \frac{\partial \delta w(t,\xi)}{\partial \xi} d\xi + Z\left[\frac{\partial w(t,s)}{\partial s} + \frac{\partial w_0(s)}{\partial s}\right] \frac{\partial \delta w(t,s)}{\partial s}\Bigg|_{s=l(t)} \right\} dZ.$$

If the variation of the deflection $\delta w(t, s)$ satisfies the conditions $\delta w(t, s) = \partial \delta w(t, s)/\partial s = 0$ at $s = 0$, one can show that the variational problem under discussion is equivalent to the boundary-value problem (2.9) and (2.11).

4. Stability of a Growing Reinforced Viscoelastic Rod on a Semi-Infinite Interval. We shall assume in what follows that at each fixed time the stability condition of an elastic rod is observed. Therefore

$$1 < \lambda_1,\ t \leqslant T_0;\ 1 < \lambda_2,\ t > T_0,$$

where λ_1 and λ_2 are the minimum eigenvalues of the following boundary-value problems:

$$\frac{\partial}{\partial s}\left[EJ_{\text{red}}(t,s)\frac{\partial^2 w(t,s)}{\partial s^2}\right] + \lambda_1 N_p(t,s)\frac{\partial w(t,s)}{\partial s} = 0, \quad t \leqslant T_0,$$

$$s = 0:\ w(t,s) = \frac{\partial w(t,s)}{\partial s} = 0; \quad s = l(t):\ \frac{\partial^2 w(t,s)}{\partial s^2} = 0;$$

$$\frac{\partial}{\partial s}\left[EJ_{\text{red}}(t,s)\frac{\partial^2 w(t,s)}{\partial s^2}\right] + N_p(t,s)\frac{\partial w(t,s)}{\partial s} + \lambda_2 qZ(t)\frac{\partial w(t,s)}{\partial s} = 0, \quad t > T_0,$$

$$s = 0:\ w(t,s) = \frac{\partial w(t,s)}{\partial s} = 0; \quad s = l(t):\ EJ_{\text{red}}(t,s)\frac{\partial^2 w(t,s)}{\partial s} = q\frac{Z^2(t)}{2}\frac{\partial w(t,s)}{\partial s},$$

$EJ_{\text{red}}(t,s) = E_a J_a(s) + \int\limits_{F(t,s)} E(t-\tau^*(\rho))\, y^2 dF$ is the reduced stiffness of the transverse cross section of the rod, and $N_p(t, s)$ is the normal force created by the rod's own weight.

We shall assume that for the mechanical and geometrical characteristcs of the rod and the concentrated force $qZ(t)$ the following limiting relationships are valid ($R^0(t - \tau)$ is some difference kernel):

$$\lim_{t\to\infty} E(t-\tau^*(\rho)) = E_0, \quad \lim_{\substack{\tau\to\infty \\ t>\tau}} R(t-\tau^*(\rho), \tau-\tau^*(\rho)) = R^0(t-\tau), \tag{4.1}$$

$$\lim_{t\to\infty} l(t) = l_0, \quad \lim_{t\to\infty} F(t,s) = F_0(s),$$

$$\lim_{t\to\infty} EJ_{\text{red}}(t,s) = EJ_0(s), \quad \lim_{t\to\infty} qZ(t) = qZ_0.$$

Then using the procedure suggested in [3], one can show that in this case a growing viscoelastic rod possessing the aging property is stable on a semi-infinite time interval in the Lyapunov sense if an elastic rod whose geometrical characteristics are determined by the limiting values (4.1) is stable and the modulus of elasticity of the main material is equal to the long-term modulus of elasticity

$$E_* = E_0 - \int\limits_0^\infty R^0(\eta)\, d\eta.$$

5. Stability of a Growing Reinforced Viscoelastic Rod on a Finite Interval. The investigation of the stability of a growing viscoelastic rod on a finite time interval takes on fundamental meaning in estimating the behavior of such a rod. We note that different formulations of the problem are possible here. We shall consider two of them.

1. Let a finite time interval [0, T] be specified. It is necessary to determine the critical values of the parameters determining the growth of the rod (for example, the values of the rates which characterize the growth of the rod in the longitudinal and transverse

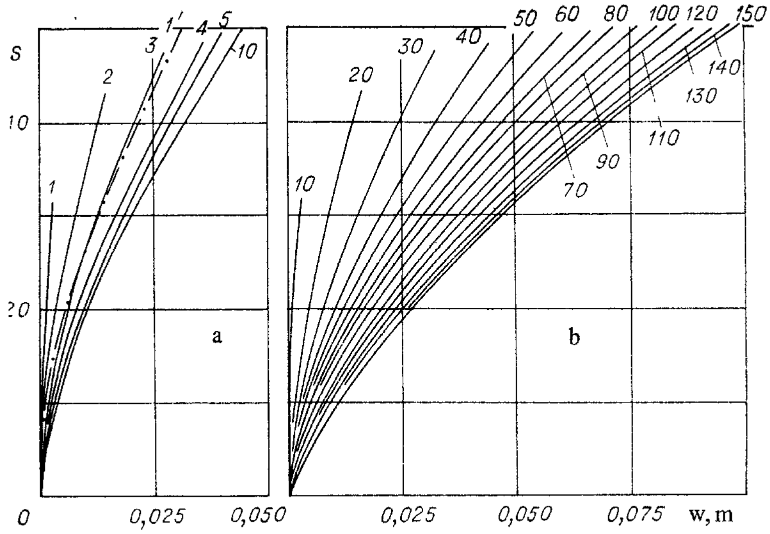

Fig. 1

directions, the variation law of the effective load in time, and so on) for which the maximum additional deflection w(t, s) does not exceed the value w* specified in advance:

$$\sup_s \sup_t |w(t, s)| \leqslant w^*, \quad t \in [0, T], \quad s \in [0, l(t)].$$

2. The value of the limiting permissible value of the deflection w* is known. It is necessary to determine the time t*, called the critical time, at which the maximum value of the rod deflection first becomes equal to w*:

$$\max_t \overline{w}(t) = w^*, \quad \overline{w}(t) = \max_s |w(t, s)|, \quad s \in [0, l(t)].$$

It is necessary for the investigation of the formulated stability problem to obtain a solution of the boundary-value problem (2.9) and (2.11) or the variational problem (3.1).

We shall consider a growing rod with transverse cross section in the form of a circular ring whose inner radius r_0 is constant and whose outer radius of an arbitrary cross section with the coordinate s varies according to the law r = r(t, s). Let the relaxation kernel of the aging viscoelastic material be [6]

$$R(t, \tau) = -\frac{\partial}{\partial \tau}\{\omega(\tau)[1 - e^{-\gamma(t-\tau)}]\}, \quad E(t) = E_0 = \text{const}, \tag{5.1}$$

where $\omega(\tau)$ is the aging function and γ is some constant.

Then we have

$$\widetilde{L}(t, \tau, s) = \mu_1(\tau, s) + \mu_2(\tau, s) e^{-\gamma(t-\tau)},$$

where

$$\mu_1(\tau, s) = \pi \int_{r_0}^{r(\tau, s)} [E_0 - \omega(\tau - \tau^*(r, s))] r^3 dr,$$

$$\mu_2(\tau, s) = \pi \int_{r_0}^{r(\tau, s)} \omega(\tau - \tau^*(r, s)) r^3 dr,$$

and $\tau^*(r, s)$ is the time of creation of a thin annular element of radius r in the cross section with coordinate s.

For relaxation kernels of the form (5.1) the integrodifferential equation (2.9) reduces to the following differential equation in partial derivatives:

$$[E_I J_I (\ddot{\varkappa} + \gamma\dot{\varkappa}) + (\mu_1 + \mu_2)\dot{\varkappa} + (\dot{\mu}_1 + \dot{\mu}_2 + \gamma\mu_1)\varkappa]' = -[N(w + w_0)']^{\cdot\cdot} - \gamma[N(w + w_0)']^{\cdot}$$

with the boundary conditions (2.11) and the initial conditions which follow from Eq. (2.9):

$$t=\tau_1^*(s):\ [(\mu_1+\mu_2)\varkappa]'=-N(w+w_0)'.$$

$$[E_{\mathrm{I}}J_{\mathrm{I}}\dot{\varkappa}+(\mu_1+\mu_2)\dot{\varkappa}+\gamma\mu\varkappa]'=-[N(w+w_0)']^{\cdot}-\gamma N(w+w_0)'.$$

A derivative with respect to the time t is denoted here by a dot, and a derivative with respect to the coordinate s is denoted by a prime.

6. Numerical Example and Analysis of the Results Obtained. Let us consider a rod of annular transverse cross section whose inner radius is kept constant along its length and whose outer radius and length vary in time according to the laws

$$r(t,s)=r_0+\frac{k}{\alpha}r_0\{1-\exp[-\alpha(t-\tau_1^*(s))]\}; \tag{6.1}$$

$$l(t)=\frac{v_0}{\alpha}(1-e^{-\alpha t}), \tag{6.2}$$

where k, α, and v_0 are some constants.

We shall take the function $\omega(\tau)$ in the form

$$\omega(\tau)=c_0+A_0e^{-\beta\tau},\ c_0,A_0 - \text{const.}$$

The additional deflection is found as a solution of the variational problem; the initial and additional deflections of the rod are represented in the form of finite summations (n is a natural number)

$$w_0(s)=\sum_{i=1}^{n}a_{i0}s^{i+1},\quad w(t,s)=\sum_{i=1}^{n}a_i(t)s^{i+1}. \tag{6.3}$$

Calculation of the integrals along the length which appear in the relationships (3.3) and (3.4) is done with the help of Simpson's formula, and the time integrals are replaced by finite summations with the help of the trapezoid formulas. As the numerical investigations have shown, one can restrict onself to keeping the first 2-3 terms in the expressions (6.3) in order to obtain satisfactory results, and one can divide the rod into 8-16 equal parts along its length. Concerning the choice of the time step, it depends both on the rate of growth of the rod and on the rheological properties of the material and should therefore be specially selected in each specific case.

The results of the solution of the problem for a rod with the following characteristics are presented in Figs. 1-5: $E_0 = 2 \times 10^4$ MPa, $A_0 = 1.5 \times 10^4$ MPa, $c_0 = 0.15 \times 10^4$ MPa, $v_0/\alpha = 50$ m, $p = 25$ kN/m^3, $q = 0$, $a_{10} = 4 \times 10^{-5}$ m, $a_{20} = \ldots = 0$, $\beta = 0.005$ day^{-1}, $r_0 = 0.25$ m, and $k = 0.005$.

It is interesting to note that the size of the deflection even of an elastic rod can depend in the limit on the nature of its growth. Thus if one considers a nongrowing rod 50 m in length with an outer diameter of 1.0 m and loaded by its own weight, the line of additional deflection is traced by curve 1' in Fig. 1a. Now we shall assume that growth of the rod occurs only in the axial direction and its outer diameter remains constant and equal to 1.0 m. In this case as the rod grows its line of additional deflection approaches the very same curve 1'. If growth of the rod occurs in accordance with the expressions (6.1) and (6.2), the lines of additional deflection of the rod at times $t' = \alpha t = 1, 2, 3, 4, 5$, and 10 have the form shown in Fig. 1a by curves 1-5 and 10, independently of the values of the parameter α ($\alpha \neq 0$, $\alpha \neq \infty$). The variation in time of the additional deflection for cross sections A and B with coordinates equal respectively to 19.6735 m (curve 1) and 43.2332 m (curve 2) with $v_0 = 5$ m/day and $\alpha = 0.1$ day^{-1} is shown in Fig. 2.

The behavior of a growing viscoelastic rod differs significantly from the behavior of a growing elastic rod. The forms of the lines of additional deflection obtained for $\alpha = 0.1$ day^{-1} and $v_0 = 5$ m/day are presented in Fig. 1b for t = 10, 20, ..., 150 days, and the variation in time of the additional deflection for the cross sections A and B is shown in Fig. 2 by curves 3 and 4. The analogous curves 5 and 6 obtained for $\alpha = 0.2$ day^{-1} and $v_0 = 10$ m/day

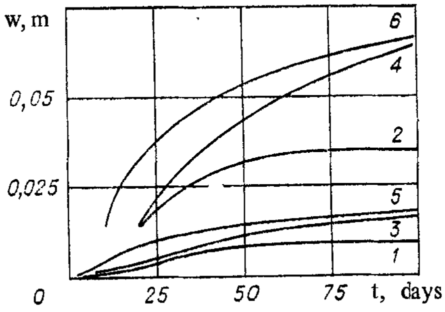

Fig. 2

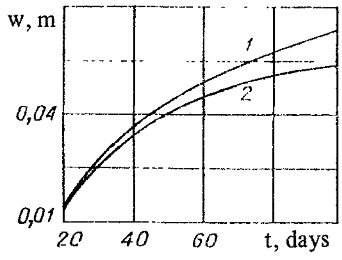

Fig. 3

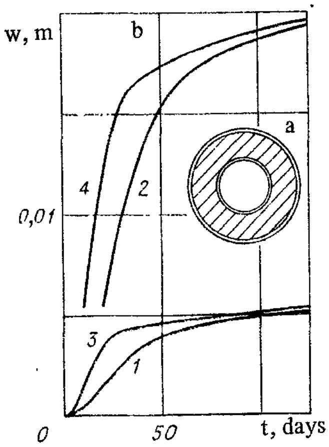

Fig. 4

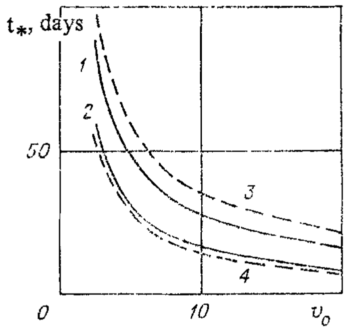

Fig. 5

are given in Fig. 2 for comparison. As is evident, the rate of growth of the rod turns out to have an important effect on the size of the deflection of the rod in each cross section.

The variation in time of the additional deflection for the cross section B with β = 0.005 and 0.020 day^{-1} (curves 1 and 2, respectively) is shown in Fig. 3 as an estimate of the effect of the aging rate of the material on the deflection characteristics of a growing rod; α = 0.1 day^{-1} and v_0 = 5 m/day. As is evident, an increase in the aging rate of the material leads to an appreciable decrease in the rod deflections.

Next we shall consider a viscoelastic rod whose growth occurs in a form which consists of two cylindrical tubes 1 cm thick each (Fig. 4a) and with diameters of the medial surface equal to 0.49 and 1.01 m, respectively. These shells are inserted one inside the other so that their longitudinal axes coincide. The space between them is filled with a viscoelastic aging material; a variation in time of the volume occupied by them occurs in accordance with the expressions (6.1) and (6.2). The shell material is elastic with a modulus of elasticity E_T = 2.1 × 10^5 MPa. One can consider a rod obtained in this way to be a reinforced rod under conditions of coupling between the shells and the viscoelastic material, for the determination of whose deflection the equations written above can be used.

The dependences between the additional deflections of the rod in the cross sections A and B and the time which correspond to growth rates α = 0.1 day^{-1}, v_0 = 5 m/day (curves 1, 2), α = 0.2 day^{-1}, v_0 = 10 m/day (curves 3, 4), and β = 0.005 day^{-1} are presented in Fig. 4b. It follows from Fig. 4b that reinforcement of the rod leads to a noticeable decrease of the rod deflections and the effect of the growth rates on the sizes of the deflections proves to be qualitatively the same as in an unreinforced growing rod.

The dependences $t_* \sim v_0$ for unreinforced (curves 1, 2) and reinforced (curves 3, 4) growing rods corresponding to values of the permissible deflection $|w|$: 0.05, 0.03, 0.02, and 0.01, respectively, are shown graphically in Fig. 5. As is evident, an increase in the growth rate of the rod leads to a significant decrease in the value of the critical time; the larger the value of $|w|$ is, the sharper is the decrease of the time t_* observed as v_0 increases.

In conclusion we shall make a remark about the determination of the initial deflection of the rod $w_0(s)$. We shall assume that as the rod grows in the axial direction its deflection along its length is measured and then the curvature for each cross section is found. In the absence of a disk on the free end of the rod the bending moment at $s = l(t)$ is equal to zero. Consequently, the curvature of the rod in this cross section is equal to the curvature $\varkappa_0(s)$, which is dependent only on the initial deflection $w_0(s)$. Knowing the curvatures $\varkappa_0(s)$ for each cross section of the rod, one can find the function $w_0(s)$ if one assumes the value of the derivative $\partial w_0(s)/\partial s$ at $s = 0$ to be known (one can set the quantity $w_0(0)$ equal to zero). In the case in which there is a rigid disk on the rod end, the value of the curvature of the rod axis on its end is found, from which the increment to the curvature produced by the bending moment created by the disk in the extreme upper cross section is subtracted. The indicated difference is evidently equal to the curvature $\varkappa_0(s)$.

As we see, the determination of the initial deflection $w_0(s)$ in a growing rod encounters some difficulties. One can overcome them by solving the following problem, which is not only of interest from the standpoint of the search for the initial deflection of a growing rod but also permits obtaining the solution of an actual engineering problem.

We shall assume that in the course of rod growth a measurement is made of its deflection in time and along its length. After completion of the rod growth (as the time segment $[0, t_1]$ runs out) the deflection continues to vary in time due to the viscous properties of the material. It is necessary to predict what value it will reach at the expiration of the time interval $[0, t]$, $t > t_1$. In order to obtain an answer to this question, one can first solve the inverse problem for the time segment $[0, t_1]$, i.e., find the deflection $w_0(s)$ from the known deflections of the rod by solving the boundary-value problem (2.9) and (2.11), and then solve the direct boundary-value problem, i.e., determine the function $w(t, s)$ from the function $w_0(s)$ found for the time interval $[0, t]$.

LITERATURE CITED

1. N. Kh. Arutyunyan, "Some problems of creep theory for nonuniformly aging bodies," Izv. Akad. Nauk SSSR, Mekh. Tverd. Tela, No. 3 (1976).
2. N. Kh. Arutyunyan and V. B. Kolmanovskii, "The stability of nonuniformly aging viscoelastic reinforced rods," Prikl. Mat. Mekh., 45, No. 6 (1981).
3. A. D. Drozdov, V. B. Kolmanovskii, and V. D. Potapov, "The stability of rods made out of a nonuniformly aging viscoelastic material," Dokl. Akad. Nauk ArmSSR, 77, No. 2 (1983).
4. N. Kh. Arutyunyan and V. D. Potapov, "The stability of a growing viscoelastic rod subjected to aging," Dokl. Akad. Nauk SSSR, 270, No. 4 (1983).
5. Yu. N. Rabotnov, Elements of Hereditary Mechanics of Solid Bodies [in Russian], Nauka, Moscow (1977).
6. N. Kh. Arutyunyan and V. V. Metlov, "Some problems of creep theory of nonuniformly aging bodies with varying boundaries," Izv. Akad. Nauk SSSR, Mekh. Tverd. Tela, No. 5 (1982).